JOURNAL OF MULTIVARIATE ANALYSIS 1, 199–214 (1971)

Series Representations of Distributions of Quadratic Form in the Normal Vectors and Generalised Variance

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Communicated by K. C. S. Pillai

Let the column vectors of $\mathbf{X} : p \times n$ be distributed as independent normals with the same covariance matrix $\mathbf{\Sigma}$. Then, the quadratic form in normal vectors is denoted by $\mathbf{XAX'} = \mathbf{S}$, where $\mathbf{A} : n \times n$ is a symmetric matrix which is assumed to be positive definite. This paper deals with the various series representations of the density function of $\mathbf{S}$ when $E(\mathbf{X}) = \mathbf{0}$, extending the idea of Kotz *et al.* (1967) to the multivariate case. Further, it gives the distribution of $|\mathbf{S}|$ when $E(\mathbf{X}) \neq \mathbf{0}$, and the results for the univariate distribution of the quadratic form in noncentral normal variates can be obtained by putting $p = 1$.

1. INTRODUCTION

Suppose that $\mathbf{x}_i : p \times 1$ $(i = 1, 2,..., n)$ are distributed normally with $E(\mathbf{x}_i) = \boldsymbol{\mu}_i$ and $\operatorname{cov}(\mathbf{x}_i, \mathbf{x}_{i'}) = \nu_{ii'}\mathbf{\Sigma}$ for $i, i' = 1, 2,..., n$, where $\mathbf{V} = (v_{ii'}) : n \times n$ and $\mathbf{\Sigma} : p \times p$ are symmetric positive definite (s.p.d.) matrices. Then, a quadratic form in normal vectors is defined by

$$\mathbf{S} = \sum_{i,i'} a_{ii'}\mathbf{x}_i\mathbf{x}_{i'}'$$

and its generalised variance by $|\mathbf{S}|$. By performing suitable linear transformations, one can easily show that the distribution of $\mathbf{S}$ is the same as that of

$$\sum_{i=1}^{n} \alpha_i \mathbf{y}_i \mathbf{y}_i', \tag{1}$$

where $\mathbf{y}_i : p \times 1$ $(i = 1, 2,..., n)$ are independent normals with $E(\mathbf{y}_i) = \boldsymbol{\delta}_i$ and $V(\mathbf{y}_i) = \mathbf{\Sigma}$, and $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n > 0$ are the characteristic (ch.) roots

Received April 30, 1970; revised December 7, 1970.

AMS 1970 subject classification: Primary 62H10; Secondary 62E15.

Key words and phrases: Distributions; quadratic forms; series representation; generalized variance.

of **VA**. It is well-known that if $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, then the distribution of **S** is noncentral Wishart and is studied by Constantine (1963). When $p = 1$, the problem is studied by various authors (see Refs. [4–5, 10–16 and 19]) in various ways. Here, we extend the unified treatment of Kotz *et al.* (1967) to the above problem when all $\boldsymbol{\delta}_i = \mathbf{0}$, and we leave the noncentral case for a later communication. We give the distribution of $|\mathbf{S}|$ for the noncentral variates and the various series representations of the density function,

$$f(\mathbf{S}) = \sum_{k=0}^{\infty} \sum_{\kappa} a_\kappa h_\kappa(\mathbf{S}), \tag{2}$$

of **S** for the central case, namely, when $\boldsymbol{\delta}_i = \mathbf{0}$ for all i.

The following types of representations are studied:

(i) $f(\mathbf{S})$ is called a combined power-series and Wishart-type representation if

$$f(\mathbf{S}) = \sum_{k=0}^{\infty} \sum_{\kappa} a_\kappa^{(1)} h_\kappa^{(1)}(\mathbf{S}), \tag{3}$$

where

$$h_\kappa^{(1)}(\mathbf{S}) = W_p(\tfrac{1}{2}n, \gamma\boldsymbol{\Sigma}^{-1}; \mathbf{S})\{(\tfrac{1}{2}n)_\kappa\}^{-1} C_\kappa(\boldsymbol{\Sigma}^{-1}\mathbf{S}), \tag{3$'$}$$

and

$$W_p(\tfrac{1}{2}n, \gamma\boldsymbol{\Sigma}^{-1}; \mathbf{S}) = \{\Gamma_p(\tfrac{1}{2}n)\}^{-1} \,|\, \mathbf{S} \,|^{(n-p-1)/2} \exp(-\gamma \operatorname{Tr} \boldsymbol{\Sigma}^{-1}\mathbf{S}), \qquad n > p.$$

If $\gamma = 0$, we get a power-series representation (see Hayakawa (1966)) and if $\gamma > 0$, we get the Wishart type representation (or a mixture of Wishart distributions) and for $\gamma = (1/q)$ with $q > 0$, Khatri (1966) obtained this representation.

(ii) $f(\mathbf{S})$ is called a Laguerre type representation if

$$f(\mathbf{S}) = \sum_{k=0}^{\infty} \sum_{\kappa} a_\kappa^{(2)} h_\kappa^{(2)}(\mathbf{S}), \tag{4}$$

where

$$h_\kappa^{(2)}(\mathbf{S}) = W_p(\tfrac{1}{2}n, \gamma\boldsymbol{\Sigma}^{-1}; \mathbf{S})\{(\tfrac{1}{2}n)_\kappa\}^{-1} L_\kappa^{(n-p-1)/2}(\alpha\boldsymbol{\Sigma}^{-1}\mathbf{S}), \tag{4$'$}$$

$\alpha \neq 0$. If $\alpha = \gamma > 0$, Shah (1968a) has mentioned such a representation. Here, for the first time for any $p > 1$, we show uniform convergence of such representations and calculate the bounds for

$$e_N^{(i)}(\mathbf{S}) = \left| \sum_{k=N+1}^{\infty} \sum_{\kappa} a_\kappa^{(i)} h_\kappa^{(i)}(\mathbf{S}) \right|, \qquad i = 1, 2 \tag{5}$$

for the representations mentioned above. In an abstract, Shah (1968b) mentioned that the density function of $\mathbf{S}$ for the noncentral variates can be written as a mixture of Wishart distributions but these results are not published. In (2) and (3), the usual meanings for $\Gamma_p(2^{-1}n, \kappa)$ and $L_\kappa{}^\beta(\mathbf{S})$ are assumed and are given in Constantine's paper (1966). For continuity purposes, we give these notations in the following few lines:

$$\Gamma_p(n/2, \kappa) = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma(k_i + (n - i + 1)/2) = (n/2)_\kappa \, \Gamma_p(n/2) \tag{6}$$

with $\Gamma_p(n/2) = \Gamma_p(n/2, 0)$ and $\kappa = (k_1, ..., k_p)$, $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_p \geqslant 0$ and $\sum_{i=1}^{p} k_i = k$, and

$L_\kappa{}^\beta(\mathbf{S})$ = Laguerre polynomial of degree k in the ch. roots of $\mathbf{S}$

$$= (\beta + (p+1)/2)_\kappa \, C_\kappa(\mathbf{I}) \sum_{j=0}^{k} \sum_{J} a_{\kappa,J} C_J(-\mathbf{S})/C_J(\mathbf{I}_p)(\beta + (p+1)/2)_J \tag{7}$$

and its generating function is given by

$$\sum_{k=0}^{\infty} \sum_{\kappa} L_\kappa{}^\beta(\mathbf{S}) \, C_\kappa(\boldsymbol{\theta})/k! \, C_\kappa(\mathbf{I}_p)$$
$$= |\, \mathbf{I} - \boldsymbol{\theta} \,|^{-\beta-(p+1)/2} \int_{0(p)} \exp[-Tr(\mathbf{SH\theta}(\mathbf{I} - \boldsymbol{\theta})^{-1}\mathbf{H}'] \, d\mathbf{H}, \tag{8}$$

where $C_\kappa(\mathbf{S})$ is the zonal polynomial of degree k in the ch. roots of $\mathbf{S}$ and the coefficients $a_{\kappa,J}$ in (7) satisfy the relations

$$\sum_{j=0}^{k} \sum_{J} a_{\kappa,J} C_J(\mathbf{S})/C_J(I) = C_\kappa(\mathbf{I} + \mathbf{S}) \qquad \text{and} \qquad a_{\kappa,J} \geqslant 0.$$

The integration in (8) is over an orthogonal group $0(p)$, $\mathbf{H} : p \times p$ is orthogonal and norm of $\boldsymbol{\theta} = \|\, \boldsymbol{\theta} \,\| < 1$.

2. Series Representations of a Density Function of S in the Central Case

2.1. *Some Preliminary Results*

Lemma 1. *Let* $\mathbf{A}$ *be a symmetric matrix with* a_i $(i = 1, 2, ..., p)$ *as the* ch. *roots of* $\mathbf{A}$. *If* $\mathbf{A}_0 = \operatorname{diag}(|\, a_1 \,|, ..., |\, a_p \,|)$, *then*

$$|\, C_\kappa(\mathbf{A})| \leqslant C_\kappa(\mathbf{A}_0).$$

Proof. Let $M_\lambda(\mathbf{A})$ be the monomial symmetric function of degree $\sum_{i=1}^{p} \lambda_i$ in the ch. roots $a_1, a_2, \ldots, a_p$ of $\mathbf{A}$, where $\lambda = (\lambda_1, \ldots, \lambda_p)$, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_p \geqslant 0$, and let us write

$$C_\kappa(\mathbf{A}) = \sum_\lambda d_\lambda M_\lambda(\mathbf{A}), \qquad \kappa = (k_1, \ldots, k_p), \qquad k_1 \geqslant \cdots \geqslant k_p \geqslant 0,$$

where $\sum_\lambda$ is the summation over the partitions λ of k such that $1 \leqslant \lambda_1 \leqslant k_1$, $k = \sum_{i=1}^{p} k_i = \sum_{i=1}^{p} \lambda_i > 0$. Then, James (1968, p. 1714, 5.8) has established the following recurrence relation:

$$d_\lambda = \sum_\mu \{(\lambda_i + r) - (\lambda_j - r)\} \, d_\mu(\rho_\kappa - \rho_\lambda)^{-1}, \tag{9}$$

where

$$\rho_\lambda = \sum_{i=1}^{p} \lambda_i(\lambda_i - i), \quad \mu = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + r, \lambda_{i+1}, \ldots, \lambda_{j-1}, \lambda_j - r, \lambda_{j+1}, \ldots, \lambda_p)$$

for all r such that, when the elements of the partition μ are arranged in descending order, μ is above λ and below or equal to κ. The summation is over all such μ, including possibly, nondescending ones. We note that d_κ is given by the expression (5.14) of James (1968, p. 1714) and it is positive. Once we know $d_\kappa > 0$, we can calculate d_λ from (9). We may note that $\rho_\kappa > \rho_\lambda$ provided $\kappa \neq \lambda$, $\rho_\kappa > 0$ and $(\lambda_i + r) - (\lambda_j - r) > 0$. Hence, from (9), it is obvious that each $d_\lambda \geqslant 0$. Using this, we get

$$| C_\kappa(\mathbf{A})| \leqslant \sum_\lambda d_\lambda \, | M_\lambda(\mathbf{A})| \leqslant \sum_\lambda d_\lambda M_\lambda(\mathbf{A}_0) = C_\kappa(\mathbf{A}_0).$$

Thus, Lemma 1 is established.

LEMMA 2. *Let* $\mathbf{S} : p \times p$ *be a* p.d. *matrix and* $\boldsymbol{\theta} : q \times q$ *be a real symmetric matrix whose* ch. *roots are* $\theta_1, \theta_2, \ldots, \theta_q$ $(q > p)$ *such that* $| \omega\theta_i | < 1$ *for all* i, *where* ω *is any real or complex number. Then*

$$\sum_{k=0}^{\infty} \sum_\kappa L_\kappa^{\beta-(p+1)/2}(\mathbf{S}) \, C_\kappa(\omega\boldsymbol{\theta})/k! \, C_\kappa(\mathbf{I}_q)$$

$$= \int_{0(q)} | \mathbf{I}_p - \omega\mathbf{H}_1\boldsymbol{\theta}\mathbf{H}_1' |^{-\beta} \exp[-\omega \operatorname{Tr} \mathbf{S}\mathbf{H}_1\boldsymbol{\theta}\mathbf{H}_1'(\mathbf{I}_p - \omega\mathbf{H}_1\boldsymbol{\theta}\mathbf{H}_1')^{-1}] \, d\mathbf{H} \tag{10}$$

and

$$\left| \sum_{\kappa} L_{\kappa}^{\beta-(p+1)/2}(\mathbf{S})\, C_{\kappa}(\boldsymbol{\theta})/k!\, C_{\kappa}(\mathbf{I}_q) \right|$$

$$\leqslant \rho^{-k} \int_{0(q)} | \mathbf{I}_p - \rho \mathbf{H}_1 \boldsymbol{\theta}_0 \mathbf{H}_1' |^{-\beta} \exp[\rho \operatorname{Tr} \mathbf{S} \mathbf{H}_1 \boldsymbol{\theta}_0 \mathbf{H}_1' (\mathbf{I}_p + \mathbf{H}_1 \boldsymbol{\theta}_0 \mathbf{H}_1' \rho)^{-1}]\, d\mathbf{H}$$

$$< \rho^{-k}(1 - \rho\epsilon)^{-p\beta} \exp[\epsilon\rho(\operatorname{Tr} \mathbf{S})/(1 + \epsilon\rho)], \tag{11}$$

where β is real, the integration is over an orthogonal space $0(q)$, $\mathbf{H}' = (\mathbf{H}_1' \, \mathbf{H}_2')$ *is an orthogonal matrix*, $\mathbf{H}_1 : p \times q$, $\boldsymbol{\theta}_0 = \operatorname{diag}(| \theta_1 |,..., | \theta_p |)$, $\epsilon = \max_i | \theta_i |$ *and* ρ *is any number such that* $0 < \rho\epsilon < 1$.

Proof. Let $\mathbf{H} : q \times q$ be an orthogonal matrix. Then, using James' result (1964, Eq. (23)), we have,

$$C_{\kappa}(\mathbf{I}_p)\, C_{\kappa}(\boldsymbol{\theta})/C_{\kappa}(\mathbf{I}_q) = \int_{0(q)} C_{\kappa}(\mathbf{H}_1 \boldsymbol{\theta} \mathbf{H}_1')\, d\mathbf{H}.$$

Hence, the left side of (10) becomes

$$\int_{0(q)} \left[\sum_{k=0}^{\infty} \sum_{\kappa} L_{\kappa}^{\beta-(p+1)/2}(\mathbf{S})\, C_{\kappa}(\omega \mathbf{H}_1 \boldsymbol{\theta} \mathbf{H}_1')/k!\, C_{\kappa}(\mathbf{I}_p) \right] d\mathbf{H}$$

and then using (8), we have

$$\int_{0(q)} \int_{0(p)} | \mathbf{I}_p - \mathbf{H}_3 \mathbf{H}_1 \boldsymbol{\theta} \mathbf{H}_1' \mathbf{H}_3' \omega |^{-\beta}$$
$$\times \exp[-\operatorname{Tr} \mathbf{S} \mathbf{H}_3 \mathbf{H}_1 \boldsymbol{\theta} \mathbf{H}_1' \omega (\mathbf{I}_p - \mathbf{H}_1 \boldsymbol{\theta} \mathbf{H}_1' \omega)^{-1} \mathbf{H}_3']\, d\mathbf{H}_3\, d\mathbf{H}.$$

Transforming $\mathbf{H}$ by the relation

$$\begin{pmatrix} \mathbf{H}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{q-p} \end{pmatrix} \mathbf{H} = \mathbf{H}_4$$

with $d\mathbf{H} = d\mathbf{H}_4$ and then integrating over $\mathbf{H}_3$, we get the left side of (10) by replacing $\mathbf{H}_4$ by $(\mathbf{H}_1' \, \mathbf{H}_2') = \mathbf{H}$. (11) follows from the well-known Cauchy's inequality. Thus, Lemma 2 is established.

LEMMA 3. *Let* $\mathbf{Z} : p \times p$ *be a complex symmetric matrix such that* $\operatorname{Re}(\mathbf{Z})$ *is* p.d. *where* $\mathrm{Re}(\mathbf{Z})$ = *real part of* $\mathbf{Z}$. *Then, the Laplace transform of the density function of* $\mathbf{S}$ *is given by*

$$E\exp(-\mathrm{Tr}\,\mathbf{ZS}) = \prod_{j=1}^{n} |\mathbf{I}_p + 2\alpha_j \mathbf{\Sigma Z}|^{-1/2} \exp\left[-\sum_{j=1}^{n} \alpha_j \boldsymbol{\delta}_j' \mathbf{Z}(\mathbf{I}_p + 2\alpha_j \mathbf{\Sigma Z})^{-1} \boldsymbol{\delta}_j\right],$$

where α_j*'s and* $\boldsymbol{\delta}_j$*'s are defined in* (1).

This follows directly from the known results on normal distribution.

Lemma 4. *Let* $\{h_\kappa\}$ *be a sequence of measurable complex-valued functions on the space of* p.d. *matrices and let* $\{a_\kappa\}$ *be a sequence of complex numbers such that*

$$\sum_{k=0}^{\infty} \left| \sum_{\kappa} a_\kappa h_\kappa(\mathbf{S}) \right| < a \exp(\mathrm{Tr}\,\mathbf{BS}) \qquad \textit{for almost all } \mathbf{S} > \mathbf{0}, \tag{12}$$

where a *is a real number and* $\mathbf{B}$ *is a symmetric matrix. Define*

$$f(\mathbf{S}) = \sum_{k=0}^{\infty} \sum_{\kappa} a_\kappa h_\kappa(\mathbf{S}) \qquad (\textit{well defined } \text{a.e.} \textit{ for } \mathbf{S} > \mathbf{0}).$$

Then, the Laplace transforms $\hat{h}_\kappa(\mathbf{Z})$ *and* $\hat{f}(\mathbf{Z})$ *of* $h_\kappa(\mathbf{S})$ *and* $f(\mathbf{S})$, *respectively, exist for* $\mathrm{Re}(\mathbf{Z})$ (*i.e., Real part of* $\mathbf{Z}$) $> \mathbf{B}$, *and*

$$\hat{f}(\mathbf{Z}) = \sum_{k=0}^{\infty} \sum_{\kappa} a_\kappa \hat{h}_\kappa(\mathbf{Z}) \qquad \text{for} \quad \mathrm{Re}(\mathbf{Z}) > \mathbf{B}.$$

The proof is an immediate consequence of Lebesgue's dominated convergence theorem.

2.2. *Method of Approach*

For the series representations, we shall take

$$\hat{h}_\kappa(\mathbf{Z}) = \xi(\mathbf{Z})\, C_\kappa(\mathbf{G}(\mathbf{Z})), \tag{13}$$

where $\xi(\mathbf{Z})$ is nonvanishing and analytic for $\mathrm{Re}(\mathbf{Z}) > \mathbf{B}$ and $\mathbf{G}(\mathbf{Z})$ is a one-to-one mapping. The inverse mapping $\mathbf{G}^{-1}$ is defined by $\mathbf{G}^{-1}(\mathbf{G}(\mathbf{Z})) = \mathbf{Z}$. Let us take $\boldsymbol{\theta} = \mathbf{G}(\mathbf{Z})$. Then $\mathbf{Z} = \mathbf{G}^{-1}(\boldsymbol{\theta})$. Moreover, let us define

$$M(\boldsymbol{\theta}) = L_0(\mathbf{G}^{-1}(\boldsymbol{\theta}))/\xi(\mathbf{G}^{-1}(\boldsymbol{\theta})), \tag{14}$$

where $L_0(\mathbf{Z}) = E\exp(-\mathrm{Tr}\,\mathbf{ZS})$ when $\boldsymbol{\delta}_i = \mathbf{0}$ for all $i = 1, 2, \ldots, n$. If we take $f(\mathbf{S})$ as the density function of $\mathbf{S}$ when all $\boldsymbol{\delta}_i$'s are zero, and $\hat{f}(\mathbf{Z})$ the Laplace transform of $f(\mathbf{S})$, then

$$\hat{f}(\mathbf{Z}) = L_0(\mathbf{Z})$$

and

$$M(\boldsymbol{\theta}) = \hat{f}(\mathbf{G}^{-1}(\boldsymbol{\theta}))/\xi(\mathbf{G}^{-1}(\boldsymbol{\theta})) = \sum_{k=0}^{\infty} \sum_{\kappa} a_\kappa C_\kappa(\boldsymbol{\theta}). \tag{15}$$

From (15), we can find a_κ as the coefficient of $C_\kappa(\boldsymbol{\theta})$ in the expansion of $M(\boldsymbol{\theta})$. Using these coefficients in $f(\mathbf{S})$, if it satisfies the conditions of Lemma 4, then by uniqueness of the Laplace transform, we get the required representation of the density function of $\mathbf{S}$. These conditions are verified for all the series representations and error terms are calculated.

2.3. *Combined Power Series and Wishart Type representation*

Let us take

$$f_1(\mathbf{S}) = \sum_{k=0}^{\infty} \sum_{\kappa} a_\kappa^{(1)} h_\kappa^{(1)}(\mathbf{S}), \tag{16}$$

where $h_\kappa^{(1)}(\mathbf{S})$ is defined by (3′). From the known results (see Constantine (1963)), we get

$$\hat{h}_\kappa^{(1)}(\mathbf{Z}) = | \gamma \mathbf{I} + \boldsymbol{\Sigma}\mathbf{Z} |^{-n/2} | \boldsymbol{\Sigma} |^{n/2} C_\kappa((\gamma \mathbf{I} + \boldsymbol{\Sigma}\mathbf{Z})^{-1})$$

provided $\mathrm{Re}(\gamma \boldsymbol{\Sigma}^{-1} + \mathbf{Z}) > \mathbf{0}$. Hence, we have

$$\xi(\mathbf{Z}) = | \gamma \mathbf{I} + \boldsymbol{\Sigma}\mathbf{Z} |^{-n/2} | \boldsymbol{\Sigma} |^{n/2} \qquad \text{and} \qquad \mathbf{G}(\mathbf{Z}) = (\gamma \mathbf{I} + \boldsymbol{\Sigma}\mathbf{Z})^{-1} = \boldsymbol{\theta}.$$

Note that $\mathbf{G}(\mathbf{Z})$ is a one-to-one mapping. Hence, using (15), we have

$$\begin{aligned} M(\boldsymbol{\theta}) &= \hat{f}_1(\mathbf{G}^{-1}(\boldsymbol{\theta}))/\xi(\mathbf{G}^{-1}(\boldsymbol{\theta})) = \sum_{k=0}^{\infty} \sum_{\kappa} a_\kappa^{(1)} C_\kappa(\boldsymbol{\theta}) \\ &= a_0^{(1)} \prod_{j=1}^{n} | \mathbf{I} - \beta_j \boldsymbol{\theta} |^{-n/2} \\ &= a_0^{(1)} \sum_{k=0}^{\infty} \sum_{\kappa} (n/2)_\kappa\, C_\kappa(\mathbf{A}_1)\, C_\kappa(\boldsymbol{\theta})/k!\, C_\kappa(\mathbf{I}_n), \end{aligned} \tag{17}$$

where $a_0^{(1)} = | \mathbf{A} |^{-p/2} | 2\boldsymbol{\Sigma} |^{-n/2}$, $\mathbf{A} = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$, $\mathbf{A}_1 = \mathrm{diag}(\beta_1, \ldots, \beta_n)$, and $\beta_j = (2\alpha_j\gamma - 1)/(2\alpha_j)$. In the expansion of the determinant occurring in (17), we have used the corresponding result established by Khatri (1966), and the expansion is valid if and only if

$$\max_i | \mathrm{ch.}_i(\boldsymbol{\theta})| < 1/\epsilon \qquad \text{or} \qquad \min_i | \mathrm{ch.}_i(\boldsymbol{\Sigma}\mathbf{Z}) + \gamma | > \epsilon, \tag{18}$$

where

$$\epsilon = \max_j |\beta_j| = \max_j |\gamma - (2\alpha_j)^{-1}|.$$

Then, from (17), we get

$$a_\kappa^{(1)} = a_0^{(1)}(n/2)_\kappa\, C_\kappa(\mathbf{A}_1)/k!\, C_\kappa(I_n). \tag{19}$$

Using (19) and (3′) in (16), we get

$$f_1(\mathbf{S}) = a_0^{(1)} \sum_{k=0}^{\infty} \sum_{\kappa} C_\kappa(\mathbf{A}_1)\, C_\kappa(\mathbf{\Sigma}^{-1}\mathbf{S})\{k!\, C_\kappa(\mathbf{I}_n)\}^{-1}\, W_p(n/2, \gamma\mathbf{\Sigma}^{-1}; \mathbf{S}), \tag{20}$$

where γ is a real number. In order that $f_1(\mathbf{S})$ is the density function of $\mathbf{S}$, it should satisfy the conditions of Lemma 4. It is easy to see that

$$\sum_{k=0}^{\infty} \left| \sum_{\kappa} a_\kappa^{(1)} h_\kappa^{(1)}(\mathbf{S}) \right| < a_0^{(1)} W_p(n/2, (\gamma - \epsilon)\, \mathbf{\Sigma}^{-1}; \mathbf{S}).$$

Since $\mathrm{Re}(\mathbf{Z}) > (-\gamma + \epsilon)\, \mathbf{\Sigma}^{-1}$ satisfies the condition (18), we get $f_1(\mathbf{S})$ as the density function of $\mathbf{S}$ by using the uniqueness property of Laplace transform and Lemma 4. The series (20) will be uniformly convergent if $\gamma > \epsilon$. For the rapidity of convergence and the choice of γ, we find the upper bound for

$$e_N^{(1)}(\mathbf{S}) = \left| \sum_{k=N+1}^{\infty} \sum_{\kappa} a_\kappa^{(1)} h_\kappa^{(1)}(\mathbf{S}) \right|.$$

Let us define $\mathbf{Y} : p \times n$ a random matrix and a domain

$$D = D\{\mathbf{Y} \mid \mathbf{S} = \mathbf{YY}'\}.$$

Then, it is easy to prove that

$$k!\, a_\kappa^{(1)} h_\kappa^{(1)}(\mathbf{S})$$
$$= \pi^{-pn/2} a_0^{(1)} \int_D \int_{0(n)} \exp(-\gamma \operatorname{Tr} \mathbf{\Sigma}^{-1}\mathbf{YY}')\, C_\kappa[\mathbf{\Sigma}^{-1}\mathbf{YHA}_1\mathbf{H}'\mathbf{Y}']\, d\mathbf{H}\, d\mathbf{Y}. \tag{21}$$

Then

$$e_N^{(1)}(\mathbf{S}) = a_0^{(1)} \pi^{-pn/2} \left| \int_D \int_{0(n)} \left[\sum_{k=N+1}^{\infty} \{\operatorname{Tr}(\mathbf{\Sigma}^{-1}\mathbf{YHA}_1\mathbf{H}'\mathbf{Y}')\}^k/k! \right] \times \exp(-\gamma \operatorname{Tr} \mathbf{\Sigma}^{-1}\mathbf{YY}')\, d\mathbf{H}\, d\mathbf{Y} \right|. \tag{22}$$

Case 1. Let $\mathbf{A}_1$ be negative semidefinite, i.e.,

$$\gamma \leqslant (2\alpha_1)^{-1} \qquad \text{where} \qquad \alpha_1 > \cdots > \alpha_n > 0.$$

When $\gamma = 0$, we get results for power-series expansion. Noting

$$\sum_{k=0}^{2r} (-x)^k/k! \geqslant \exp(-x) \geqslant \sum_{k=0}^{2r+1} (-x)^k/k!,$$

and so

$$\left| \sum_{k=N+1}^{\infty} (-x)^k/k! \right| = \left| \exp(-x) - \sum_{k=0}^{N} (-x)^k! \right| < x^{N+1}/(N+1)!$$

for $x \geqslant 0$ and r and N positive integers; we get after some simplifications the final result as

$$\begin{aligned} e_N^{(1)}(\mathbf{S}) &\leqslant a_0^{(1)} W_p(n/2, \gamma\mathbf{\Sigma}^{-1}; \mathbf{S}) \sum_{\eta} C_\eta(-\mathbf{A}_1)\, C_\eta(\mathbf{\Sigma}^{-1}\mathbf{S})/C_\eta(I_n)(N+1)! \\ &\leqslant a_0^{(1)} W_p(n/2, \gamma\mathbf{\Sigma}^{-1}; \mathbf{S})\, \epsilon^{N+1}(\mathrm{Tr}\, \mathbf{\Sigma}^{-1}\mathbf{S})^{N+1}/(N+1)! \end{aligned} \tag{23}$$

where $\sum_\eta$ is the summation over the partitions of $(N+1)$ and $\epsilon = (2\alpha_n)^{-1} - \gamma$. We note that power-series expansion does not converge rapidly and uniformly, because for this case the uniform convergence has the condition

$$(4\alpha_n)^{-1} < \gamma \leqslant (2\alpha_1)^{-1}.$$

The best choice of γ is $\gamma = (2\alpha_1)^{-1}$, when $\mathbf{A}_1$ is negative (semi-) definite, and $\epsilon = (\alpha_n^{-1} - \alpha_1^{-1})/2$.

Case 2. Let $\gamma > (2\alpha_1)^{-1}$. A_1 may have some negative elements. Let us define $A_2 = \mathrm{diag}(|\beta_1|, |\beta_2|, \ldots, |\beta_p|)$. Then, it is easy to see that

$$\begin{aligned} &\sum_{k=N+1}^{\infty} [\mathrm{Tr}(\mathbf{\Sigma}^{-1}\mathbf{YHA}_1\mathbf{H}'\mathbf{Y}')]^k/k! \\ &\qquad \leqslant [\mathrm{Tr}(\mathbf{\Sigma}^{-1}\mathbf{YHA}_2\mathbf{H}'\mathbf{Y}')]^{N+1}\{(N+1)!\}^{-1} \exp(\epsilon\, \mathrm{Tr}\, \mathbf{\Sigma}^{-1}\mathbf{YY}') \end{aligned}$$

and hence (22) gives

$$\begin{aligned} e_N^{(1)}(\mathbf{S}) &\leqslant a_0^{(1)} W_p(n/2, (\gamma - \epsilon)\, \mathbf{\Sigma}^{-1}; \mathbf{S}) \sum_{\eta} C_\eta(\mathbf{A}_2)\, C_\eta(\mathbf{\Sigma}^{-1}\mathbf{S})/C_\eta(\mathbf{I}_n)(N+1)! \\ &\leqslant a_0^{(1)} W_p(n/2, (\gamma - \epsilon)\, \mathbf{\Sigma}^{-1}; \mathrm{S})\epsilon^{N+1}(\mathrm{Tr}\, \mathbf{\Sigma}^{-1}\mathbf{S})^{N+1}/(N+1)! \end{aligned} \tag{24}$$

Note that (20) is uniformly convergent if $\gamma > \epsilon$ and this gives us the condition that $\gamma > (4\alpha_n)^{-1}$. The best choice of ϵ will be

$$\epsilon_0 = \inf_{\gamma} \max_{j} | \gamma - (2\alpha_j)^{-1} |.$$

We note that the solution is $\gamma = (\alpha_1^{-1} + \alpha_n^{-1})/4$ and hence

$$\epsilon = (\alpha_1 - \alpha_n)/4\alpha_1\alpha_n \qquad \text{and} \qquad \gamma - \epsilon_0 = (2\alpha_n)^{-1}.$$

This shows that $\gamma = (\alpha_1^{-1} + \alpha_n^{-1})/4$ is better than the best choice given in Case 1.

2.4. *Laguerre-type Representation*

Let us take

$$f_2(\mathbf{S}) = \sum_{k=0}^{\infty} \sum_{\kappa} a_\kappa^{(2)} h_\kappa^{(2)}(\mathbf{S}), \tag{25}$$

where $h_\kappa^{(2)}(\mathbf{S})$ is given by (4′). Using the results of Constantine (1966), we get

$$\hat{h}_\kappa^{(2)}(\mathbf{Z}) = | \mathbf{I}\gamma + \mathbf{\Sigma Z} |^{-n/2} C_\kappa[\mathbf{I} - \alpha(\gamma\mathbf{I} + \mathbf{\Sigma Z})^{-1}] \, | \mathbf{\Sigma} |^{n/2}$$

if $\operatorname{Re}(\mathbf{\Sigma Z}) > -\gamma\mathbf{I}$. Here, we have

$$\xi(\mathbf{Z}) = | \gamma\mathbf{I} + \mathbf{\Sigma Z} |^{-n/2} \, | \mathbf{\Sigma} |^{n/2} \qquad \text{and} \qquad \mathbf{G}(\mathbf{Z}) = \mathbf{I} - \alpha(\gamma\mathbf{I} + \mathbf{\Sigma Z})^{-1} = \boldsymbol{\theta}.$$

Since $\mathbf{G}$ is a one-to-one mapping, we must have $\alpha \neq 0$. With the above results, we have

$$\begin{aligned} \sum_{k=0}^{\infty} \sum_{\kappa} a_\kappa^{(2)} C_\kappa(\boldsymbol{\theta}) &= \hat{f}_2(\mathbf{G}^{-1}(\boldsymbol{\theta}))/\xi(\mathbf{G}^{-1}(\boldsymbol{\theta})) \\ &= a_0^{(2)} \prod_{j=1}^{n} | \mathbf{I} - \phi_j \boldsymbol{\theta} |^{-1/2} \\ &= a_0^{(2)} \sum_{k=0}^{\infty} \sum_{\kappa} (n/2)_\kappa \, C_\kappa(\mathbf{A}_3) \, C_\kappa(\boldsymbol{\theta})/k! \, C_\kappa(\mathbf{I}_n), \end{aligned} \tag{26}$$

where $\phi_j = (1 - 2\alpha_j\gamma)/(1 - 2\alpha_j\gamma + 2\alpha_j\alpha)$, $\mathbf{A}_3 = \operatorname{diag}(\phi_1, \phi_2, \ldots, \phi_n)$ and $a_0^{(2)} = | 2\mathbf{\Sigma} |^{-n/2} \, | \mathbf{I}_n - \mathbf{A}_3 |^{p/2} \, | \mathbf{A} |^{-p/2}$ with $\mathbf{A} = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$. The expansion on the right side of (26) is valid if and only if

$$\max_i | \operatorname{ch}._i(\boldsymbol{\theta})| < \epsilon_1^{-1} \qquad \text{or} \qquad \max_i | 1 - \alpha/(\gamma + \operatorname{ch}._i(\mathbf{\Sigma Z})| < \epsilon_1^{-1} \tag{27}$$

where $\epsilon_1 = \max_j | \phi_j |$. Then, from (26), we have

$$a_\kappa^{(2)} = a_0^{(2)} (n/2)_\kappa \, C_\kappa(\mathbf{A}_3)/k! \, C_\kappa(\mathbf{I}_n). \tag{28}$$

Using (28) and (4′) in (25), we get

$$f_2(\mathbf{S}) = a_0^{(2)} W_p(n/2, \mathbf{\Sigma}^{-1}\gamma; \mathbf{S}) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{L_\kappa^{(n-p-1)/2}(\alpha\mathbf{\Sigma}^{-1}\mathbf{S})\, C_\kappa(\mathbf{A}_3)}{k!\, C_\kappa(\mathbf{I}_n)}. \tag{29}$$

The use of Lemma 2 (see equation (10)) shows that the series (29) is convergent if and only if $\epsilon_1 < 1$. Hence, we shall choose our γ and α so that $\epsilon_1 < 1$. In order to see that the conditions of Lemma 4 are satisfied for $f_2(\mathbf{S})$ of (29), we use Lemma 2 and (Eq. (11)) and taking $\alpha > 0$, obtain

$$\sum_{k=0}^{\infty} \Big| \sum_{\kappa} a_\kappa^{(2)} h_\kappa^{(2)}(\mathbf{S}) \Big| \\ \leqslant a_0^{(2)} W_p(n/2, (\gamma - \alpha\rho/(1+\rho))\, \mathbf{\Sigma}^{-1}; \mathbf{S})(1-\rho)^{-np/2}(1-\epsilon_1/\rho)^{-1}$$

provided we choose ρ to satisfy $\epsilon_1 < \rho < 1$.

If $\mathrm{Re}(\mathbf{Z}) > (-\gamma + \alpha\rho/(1+\rho))\, \mathbf{\Sigma}^{-1}$; then it is easy to see that (27) is satisfied. Hence, by the uniqueness of Laplace transform and Lemma 4, $f_2(\mathbf{S})$ becomes the density function of $\mathbf{S}$. Moreover, if $\gamma > \alpha\rho/(1+\rho)$, then $f_2(\mathbf{S})$ is uniformly continuous over the space of $\mathbf{S} > \mathbf{0}$.

Now, using Lemma 2, we see that if

$$e_N^{(2)}(\mathbf{S}) = \Big| \sum_{k=N+1}^{\infty} \sum_{\kappa} a_\kappa^{(2)} h_\kappa^{(2)}(\mathbf{S}) \Big|,$$

then

$$e_N^{(2)}(\mathbf{S}) \leqslant a_0^{(2)} \inf_{\epsilon_1<\rho<1} \{W_p(n/2, (\gamma - \alpha\rho/(1+\rho))\, \mathbf{\Sigma}^{-1}; \mathbf{S}) \\ (\epsilon_1/\rho)^{N+1}(1-\epsilon_1/\rho)^{-1}(1-\rho)^{-pn/2}\}/(N+1)! \tag{30}$$

A choice of ρ which makes the formula takes a simple form (but which may not be the best from other points of view) is $\rho = \epsilon_1^{1/2}$. Then, (30) can be rewritten as

$$e_N^{(2)}(\mathbf{S}) \leqslant a_0^{(2)} W_p(n/2, (\gamma - \alpha^0/2)\, \mathbf{\Sigma}^{-1}; \mathbf{S})\, \epsilon_1^{(N+1)/2}(1-\sqrt{\epsilon_1})^{-(pn+2)/2}/(N+1)! \tag{31}$$

where $\alpha^0 = \alpha\sqrt{\epsilon_1}/(1+\sqrt{\epsilon_1})$. First of all, we note that we have taken $\alpha > 0$. Since $\epsilon_1 < 1$, we get, $1 - (1 - 2\alpha_j\gamma)/(1 - 2\alpha_j\gamma + 2\alpha_j\alpha) > 0$ for all j and this means that α and $(1 - 2\alpha_j\gamma + 2\alpha_j\alpha)$ must have the same sign. Because γ is any real number, we can always take $\alpha > 0$. It is easy to verify that if $\gamma_1 = 1/\gamma$ and $\alpha_0 = \alpha/\gamma$,

$$\epsilon_1 = \max[(\gamma_1 - 2\alpha_n)/(\gamma_1 - 2\alpha_n + 2\alpha_n\alpha_0), (2\alpha_1 - \gamma_1)/(\gamma_1 - 2\alpha_1 + 2\alpha_1\alpha_0)]. \tag{32}$$

If we want to improve the bound (31) over the one obtained when $\alpha_0 = 1$, we must satisfy the following two conditions for some γ_1 and α_0 :

$$(\alpha_1 + \alpha_n)(2 - \alpha_0) > \gamma_1 \tag{33}$$

and

$$\epsilon_1 < (\alpha_1 - \alpha_n)/(\alpha_1 + \alpha_n). \tag{34}$$

It has been found that if (33) is satisfied, then (34) is not satisfied and, conversely, but $\gamma_1 = (\alpha_1 + \alpha_n)(2 - \alpha_0)$ and $\epsilon_1 = (\alpha_1 - \alpha_n)/(\alpha_1 + \alpha_n)$ are satisfied at $\alpha_0 = 1$. Thus, introduction of α may or may not improve the results. The optimum choice of ϵ_1 can be obtained satisfying only one condition (33) or (34).

Further, if we take $\gamma_1 = \alpha_1 + \alpha_n$, then,

$$\epsilon_1 = (\alpha_1 - \alpha_n)/\{(\alpha_1 + \alpha_n) + 2\alpha_n(\alpha_0 - 1)\}.$$

Choose α_0 so that

$$1 - \alpha_0 \sqrt{\epsilon_1}/(1 + \sqrt{\epsilon_1}) = 1/2.$$

From this, we find that

$$\alpha_0 = 1 + \alpha_n/2(\alpha_1 - \alpha_n)$$

and hence

$$\epsilon_1 = (1 - \alpha_n/\alpha_1)^2.$$

We notice that these values of ϵ_1 , α_0 and γ_1 simplify (31) still further to

$$\begin{aligned} e_N^{(2)}(\mathbf{S}) \leqslant{} & | 2\boldsymbol{\Sigma} |^{-n/2}(\alpha_1/\alpha_n)(\alpha_n^{-1} - \alpha_1^{-1})^{pn/2}[(\alpha_1 - \alpha_n)/\alpha_1]^{N+1}\{(N + 1)!\}^{-1} \\ & \times W_p(n/2, \boldsymbol{\Sigma}^{-1}/2(\alpha_1 + \alpha_n); \mathbf{S}). \end{aligned} \tag{35}$$

3. Distribution of $| \mathbf{S} | = \omega$

Lemma 5. *Let* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{S}$ *be* $p \times p$ *symmetric matrices. Then,*

$$\int_{0(p)} C_\kappa(\mathbf{SHAH'})\, C_\eta(\mathbf{SHBH'})\, d\mathbf{H} = \sum_\nu C_\nu(\mathbf{S})\, P^\nu_{\kappa,\eta}(\mathbf{A}, \mathbf{B}), \tag{36}$$

where the summation is over all partitions ν *of* $k + n$, κ *and* η *are, respectively, given partitions of* k *and* n, *and* $P^\nu_{\kappa,\eta}(\mathbf{A}, \mathbf{B})$ *depends on* $\mathbf{A}$ *and* $\mathbf{B}$ *only.*

Proof. On account of symmetric property of zonal polynomials, we can write,

$$C_\kappa(\mathbf{Q}) = \sum_\lambda d_{\lambda,\kappa} \prod_{i=1}^{p} (\mathrm{Tr}_i(\mathbf{Q}))^{\lambda_i},$$

where the summation is over λ_i's such that $\sum_{i=1}^{p} i\lambda_i = k$, $\mathrm{Tr}_i(\mathbf{Q})$ is the sum of principal minors of order $i(=1, 2,..., p)$ and $d_{\lambda,\kappa}$ are constants depending on the partition κ. Further, without loss of generality, we can take $\mathbf{S}$ to be a diagonal matrix, and

$$\mathrm{Tr}_i(\mathbf{SQ}) = \sum_{\substack{j_1, j_2, ..., j_i \\ (j_1 \geqslant j_2 \geqslant ... \geqslant j_i)}} (s_{j_1} s_{j_2} \cdots s_{j_i}) |(\mathbf{Q})_{j_1, ..., j_i}|$$

where $(\mathbf{Q})_{j_1,...,j_i}$ is the submatrix formed by $j_1, ..., j_i$ rows and $j_1, ..., j_i$ columns of $\mathbf{Q}$. From the product $C_\kappa(\mathbf{SHAH}')\, C_\eta(\mathbf{SHBH}')$, we can collect the coefficient of $s_1^{t_1} s_2^{t_2} \cdots s_p^{t_p}$ and integrating over $\mathbf{H}$ gives the same value for any permutations of $t_1, t_2, ..., t_p$. Thus, Lemma 5 is established.

LEMMA 6. *Let* $\mathbf{Y}$ *and* $\mathbf{T}$ *be matrices of order* $p \times n$, $n \geqslant p$ *and let* $\mathbf{A} : n \times n$ *be a symmetric matrix. Then for* $\mathbf{H} \in O(n)$ *and* $\mathbf{H}_1 \in O(p)$, *there exists a function* $R_\kappa(\mathbf{A}, \mathbf{T}'\mathbf{T})$ *such that*

$$\int_{O(p)} \int_{O(n)} \exp[\mathrm{Tr}(\mathbf{SHAH}'\mathbf{S}' + 2\mathbf{H}_1\mathbf{SHT}')]\, d\mathbf{H}_1\, d\mathbf{H}$$
$$= \sum_{k=0}^{\infty} \sum_\kappa R_\kappa(\mathbf{A}, \mathbf{T}'\mathbf{T})\, C_\kappa(\mathbf{SS}')/k!\, C_\kappa(\mathbf{I}_p). \tag{37}$$

This follows from Lemma 5 and

$$\int_{O(p)} \exp(2\, \mathrm{Tr}\, \mathbf{XH}_1')\, d\mathbf{H}_1 = {}_0F_1(p/2; \mathbf{XX}').$$

We note that when $\mathbf{A} = -\mathbf{B}$, $\mathbf{B}$ is p.d. and $\mathbf{T} = \mathbf{T}_1\mathbf{B}^{1/2}$, where $\mathbf{B}^{1/2}$ is a symmetric square root of $\mathbf{B}$, then comparing (37) and Theorem 7 of Hayakawa (1969, p. 17), we get

$$R_\kappa(-\mathbf{B}, \mathbf{B}^{1/2}\mathbf{T}_1'\mathbf{T}_1\mathbf{B}^{1/2}) = P_\kappa(\mathbf{T}_1, \mathbf{B})/(n/2)_\kappa, \tag{38}$$

where $P_\kappa(\mathbf{T}_1, \mathbf{B})$ is defined by Hayakawa (1969, p. 17) as

$$\exp(-\mathrm{Tr}\, \mathbf{T}_1\mathbf{T}_1')\, P_\kappa(\mathbf{T}_1, \mathbf{B})$$
$$= (-1)^k \pi^{-pn/2} \int_{\mathbf{U}} \exp(-2\sqrt{-1}\, \mathrm{Tr}\, \mathbf{T}_1\mathbf{U}' - \mathrm{Tr}\, \mathbf{UU}')\, C_\kappa(\mathbf{UBU}')\, d\mathbf{U}. \tag{39}$$

Lemma 7. *The Mellin's transform of the density function of* $\omega = |\mathbf{S}|$, $\mathbf{S}$ *defined by* (1), *is given by*

$$\sum_{k=0}^{\infty} \sum_{\kappa} R_\kappa(\mathbf{A}_1, \boldsymbol{\delta}'\boldsymbol{\delta})\, c_0^{(1)} \prod_{i=1}^{p} \Gamma(h + k_i + (n-i+1)/2) \left| \sum/\gamma \right|^h \Big/ k!\, \gamma^k,$$

where

$$\gamma > 0, \qquad c_0^{(1)} = \left\{ \prod_{i=1}^{p} \Gamma(n-i+1)/2 \right\}^{-1} |\gamma 2\mathbf{A}|^{-p/2} \exp(-\mathrm{Tr}\, \boldsymbol{\delta}\mathbf{A}\boldsymbol{\delta}'/2),$$

$$\mathbf{A} = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n), \quad \boldsymbol{\delta} = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \ldots, \boldsymbol{\delta}_n)\, \mathbf{A}^{-1/2}, \quad \mathbf{A}_1 = \gamma\mathbf{I} - \mathbf{A}^{-1/2}.$$

Proof. After some necessary transformations, we can write the Mellin's transform of the density function of ω as

$$E(\omega^h) = c_0 |\boldsymbol{\Sigma}|^h \pi^{-pn/2} \int_D |\mathbf{YY}'|^h \exp[-\mathrm{Tr}(\mathbf{YA}^{-1}\mathbf{Y}' - 2\mathbf{Y}\boldsymbol{\delta}')/2]\, d\mathbf{Y}, \quad (40)$$

where $\boldsymbol{\delta}$ and $\mathbf{A}$ are the same as defined in Lemma 7, $D = D\{-\infty < \mathbf{Y} < \infty$ i.e., $-\infty < y_{ij} < \infty$ for all $i, j\}$ and $c_0 = |2\mathbf{A}|^{-p/2} \exp(-\mathrm{Tr}\, \boldsymbol{\delta}\mathbf{A}\boldsymbol{\delta}'/2)$. Since the domain D in (40) is invariant under the transformation $\mathbf{Y} \to \mathbf{H}_1\mathbf{YH}$ where $\mathbf{H}_1 \in 0(p)$ and $\mathbf{H} \in 0(n)$, we can introduce random orthogonal matrices and write (40) as

$$E(\omega^h) = c_0 |\boldsymbol{\Sigma}|^h \pi^{-pn/2} \int_D |\mathbf{YY}'|^h \exp(-\gamma\, \mathrm{Tr}\, \mathbf{YY}')$$
$$\times \int_{0(p)} \int_{0(n)} \exp\{\mathrm{Tr}(\mathbf{YHA}_1\mathbf{H}'\mathbf{Y}') + \mathrm{Tr}(\mathbf{H}_1\mathbf{YH}\boldsymbol{\delta}')\}\, d\mathbf{H}_1\, d\mathbf{H}\, d\mathbf{Y}. \quad (41)$$

Using Lemma 6, we get

$$E(\omega^h) = c_0 |\boldsymbol{\Sigma}|^h \sum_{k=0}^{\infty} \sum_{\kappa} R_\kappa(\mathbf{A}_1, \boldsymbol{\delta}'\boldsymbol{\delta}) \{k!\, C_\kappa(\mathbf{I}_p)\}^{-1}$$
$$\times \pi^{-pn/2} \int_D |\mathbf{YY}'|^h C_\kappa(\mathbf{YY}') \exp(-\gamma\, \mathrm{Tr}\, \mathbf{YY}')\, d\mathbf{Y}.$$

From this we get the required result.

Now, using the definition of

$$G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$$

(see Erdelyi (1953), p. 207) given by

$$G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right)$$

$$= (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} (1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} x^s \, ds,$$

we can obtain the density function $f(\omega)$ (given in Lemma 8) of ω from the inverse Mellin's formula

$$f(\omega) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} E(\omega^h)\, \omega^{-h-1} \, dh$$

for some real number c.

LEMMA 8. *In the notations of Lemma* 7, *the density function of* ω *is given by*

$$| \mathbf{\Sigma}/\gamma \,|\, c_0^{(1)} \sum_{k=0}^{\infty} \sum_{\kappa} R_\kappa(\mathbf{A}_1, \boldsymbol{\delta}'\boldsymbol{\delta})(\omega\gamma^p/| \mathbf{\Sigma} |)^{k_p+(n-p-1)/2}$$

$$\times (k!\, \gamma^k)^{-1} G_{0,p}^{p,0}(\omega\gamma^p/| \mathbf{\Sigma} | \mid k_i - k_p + (p - i)/2 \text{ for } i = 1, 2, \dots, p).$$

We note that when $p = 1$, the result is the same as the mixture of gamma distributions as given by Ruben (1962) and Kotz *et al.* (1967).

Remarks. (a) The distribution of Tr $\mathbf{S}$ where $\mathbf{S}$ is distributed as mentioned in Section 1 for the noncentral case is similar to that of a quadratic form in noncentral variates (see references) and hence it is not mentioned here. The distribution of (Tr $\mathbf{S}$) as given by Hayakawa (1969) is a power-series type and is not rapidly convergent (see for the central case, Section 2.3 of this paper and Kotz *et al.* (1967)). (b) The above results can be easily extended to complex multivariate normal distribution and so they are not explicitly given.

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